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Rate of convergence of Hermite–Padé approximants to a Nikishin-type system of analytic functions

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Abstract

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A Nikishin-type system of analytic functions is considered. In 1980, Nikishin proved that if such a system is formed by two functions and the interpolation conditions are equally distributed between them, then convergence of the corresponding Hermite–Padé approximants takes place. We recently extended this result to the case of m functions. Here, we give some estimates on the rate of convergence.

Keywords: Simultaneous approximation; Hermite–Padé approximation

1. Introduction

(1) Let $f_1, \ldots, f_m$ be a set of m formal expressions in decreasing powers of z:

$$f_i(z) = A_{k_i,i} z^{k_i} + A_{k_i-1,i} z^{k_i-1} + \cdots, \quad i = 1, \ldots, m. \tag{1}$$

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Let $r_1,\ldots,r_m$ be an arbitrary set of nonnegative integers. It is easy to verify that there exists a polynomial $Q_n(z) \neq 0$, $\deg Q_n \leqslant n = r_1 + r_2 + \cdots + r_n$, such that

$$[Q_n f_i - P_{n,i}](z) = A_i z^{-r_i-1} + \cdots, \quad i = 1,\ldots,m. \tag{2}$$

Finding Q_n reduces to solving a system of n homogeneous linear equations on the $n+1$ coefficients of Q_n. Thus a nontrivial solution can be guaranteed. In the sequel, we normalize Q_n to have 1 as leading coefficient.

It is reasonable to consider the fractions $\{P_{n,i}/Q_n\}$ as simultaneous approximations to $\{f_i\}$, $i = 1,\ldots,m$. Given n, for $m = 1$, we obtain the classical diagonal Padé approximant for which the quotient $P_{n,1}/Q_n$ defines a unique rational function. Unlike the case when $m = 1$, in general, conditions (2) do not determine a unique vector $(P_{n,1}/Q_n,\ldots,P_{n,m}/Q_n)$ of rational functions. When the set of indexes $(r_1,\ldots,r_m)$ is such that conditions (2) define a unique polynomial Q_n up to a constant factor, then this set of indexes is said to be *normal* and in that case a unique vector $(P_{n,1}/Q_n,\ldots,P_{n,m}/Q_n)$ is defined. When the system of functions is such that any set of indexes $(r_1,\ldots,r_m)$ is normal, then the system of functions is said to be *perfect*.

In [7], Nikishin introduced what is now known as a Nikishin system of m functions (see also [7] for some historical comments on Hermite–Padé approximation). For them he proved the normality of certain sets of indexes $(r_1,\ldots,r_m)$, and without use of the asymptotics of the denominators Q_n, he managed to show for $m = 2$ that if for all n, $r_1 = r_2$, then the sequence of Hermite–Padé approximants thus obtained converges uniformly to the corresponding functions in the largest possible region. Recently we proved [1] that the same is true for any Nikishin system of functions. Following some techniques exposed in [5], we wish to derive estimates on the rate with which such convergence takes place.

(2) Let (Δ_j, μ_j), $j = 1,\ldots,m$, be m pairs formed by a bounded segment $\Delta_j \subset \mathbb{R}$ and a positive finite Borel measure μ_j on Δ_j whose support contains infinitely many points. Further, $\Delta_j \cap \Delta_{j+1} = \emptyset$, $j = 1,\ldots,m-1$.

A Nikishin system $(f_1, f_2,\ldots,f_m)$ is generated by $\{(\Delta_1, \mu_1),\ldots,(\Delta_m, \mu_m)\}$ on $D = \mathbb{C}\setminus\Delta_1$ as follows (we remark that the order is important):

$$f_1(z) = \int_{\Delta_1} \frac{d\mu_1(x_1)}{z - x_1}, \qquad f_2(z) = \int_{\Delta_1} \frac{d\mu_1(x_1)}{z - x_1} \int_{\Delta_2} \frac{d\mu_2(x_2)}{x_1 - x_2},$$

$$f_3(z) = \int_{\Delta_1} \frac{d\mu_1(x_1)}{z - x_1} \int_{\Delta_2} \frac{d\mu_2(x_2)}{x_1 - x_2} \int_{\Delta_3} \frac{d\mu_3(x_3)}{x_2 - x_3}, \ldots,$$

$$f_m(z) = \int_{\Delta_1} \frac{d\mu_1(x_1)}{z - x_1} \cdots \int_{\Delta_m} \frac{d\mu_m(x_m)}{x_{m-1} - x_m}. \tag{3}$$

Note that a Nikishin system is constructed recursively. On $\mathbb{C}\setminus\Delta_j$, $j = 1,\ldots,m$, we obtain the following $m - j + 1$ functions which in turn form a Nikishin system (in the sequel, the subindex in the measure indicates the interval on which we integrate and the variable with respect to which we integrate):

$$\Delta_m: \qquad f_{m,m}(z) = \int \frac{d\mu_m}{z - x_m},$$

$$\Delta_{m-1}: \quad f_{m-1,m-1}(z)=\int\frac{\mathrm{d}\mu_{m-1}}{z-x_{m-1}}, \quad f_{m-1,m}(z)=\int\frac{f_{m,m}(x_{m-1})\,\mathrm{d}\mu_{m-1}}{z-x_{m-1}},$$

$$\vdots$$

$$\Delta_j: \qquad f_{j,i}(z)=\int\frac{f_{j+1,i}(x_j)\,\mathrm{d}\mu_j}{z-x_j}, \quad i=j,\dots,m, \quad (f_{j+1,j}\equiv 1),$$

until we arrive at Δ_1, where $f_{1,i}\equiv f_i$, $i=1,\dots,m$.

In [7] Nikishin proved several important properties concerning such systems. For example, let $(r_0, r_1,\dots,r_m)$ be a set of indexes of the form $(s, s,\dots,s)$ or $(s+1,\dots,s+1, s,\dots,s)$, $s\geqslant -1$; then for any choice of polynomials $h_i(x)$, $\deg h_i\leqslant r_i$, $i=0, 1,\dots,m$ (for $r_i=-1$ we take $h_i\equiv 1$), the function

$$[h_0+h_1f_1+\cdots+h_mf_m](x) \tag{4}$$

has at most $(r_0+1)+(r_1+1)+\cdots+(r_m+1)-1$ zeros on any fixed interval Δ disjoint from Δ_1. If $(r_0, r_1,\dots,r_m)$ is of one of the two forms described above, we say that it is *regular*.

Nikishin called any system of functions $(u_0, u_1,\dots,u_m)$ with this property on an interval Δ a *weak AT-system*. Note that on any interval Δ disjoint from Δ_j the system of functions $(1, f_{j,j},\dots,f_{j,m})$ forms a weak AT-system.

From this immediately follows (see [1, Lemma 3]) that for regular indexes (with m components) $\deg Q_n=n$ and all the zeros of Q_n are simple and lie on Δ_1 excluding the endpoints. Therefore such indexes are normal. Moreover, Nikishin asserted without proof that any set of indexes $(r_1,\dots,r_m)$ with $r_1\geqslant r_2\geqslant\cdots\geqslant r_m$ is normal.

From [1, Theorem 1] in particular the following theorem holds (there, we allowed unbounded intervals).

Theorem A. *Let* $(f_1,\dots,f_m)$ *be given by* (3), *and assume that* $\{(r_1(s),\dots,r_m(s)\}$, $s\in\mathbb{N}$, *is a sequence of multi-indexes satisfying that there exists a constant c such that for all* $s\in\mathbb{N}$ *we have* $r_i(s)\geqslant n/m-c$, $i=1,\dots,m$, $n=n(s)=r_1(s)+\cdots+r_m(s)$. *Assume that* $n\to\infty$ *as* $s\to\infty$. *Then, for every compact set* $K\subset D=\mathbb{C}\setminus\Delta_1$ *and each* $\epsilon>0$,

$$\operatorname{cap}\{z\in K: |(f_i-R_{n,i})(z)|\geqslant\epsilon\}\to 0, \quad i=1,\dots,m, \ s\to\infty, \tag{5}$$

where $\operatorname{cap}\{\cdot\}$ *denotes the logarithmic capacity of the indicated set.*

Using [3, Lemma 1], if the sequence of functions $\{R_{n,i}\}$, $s\in\mathbb{N}$, has no poles in D, then convergence in capacity yields uniform convergence on each compact subset of D. Therefore, taking into consideration the result of [7], we have the following corollary.

Corollary B. *If for each* $s\in\mathbb{N}$, $(r_1(s), r_2(s),\dots,r_m(s))$ *is regular, then*

$$R_{n,i}\to f_i, \quad K\subset D, \ s\to\infty, \ i=1,\dots,m, \tag{6}$$

(*uniformly on each compact subset of D*).

Here, we give some estimates for (5), (6). We have the following theorem (compare with [5, Theorem 1]).

Theorem 1. *If for each $s \in \mathbb{N}$, $(r_1(s), r_2(s), \ldots, r_m(s))$ is regular, then for every compact set $K \subset D$,*

$$\overline{\lim_{s\to\infty}} \| f_i - R_{n,i} \|_K^{1/2n} \leqslant e^{-\nu}, \quad \nu = \nu(K) > 0, \ i = 1, \ldots, m. \tag{7}$$

In fact, we shall prove a stronger result and give a precise value for ν.
The notations introduced above are maintained throughout the paper.

2. Proof of Theorem 1

In the proof of Theorem A we showed [1, Lemma 1] that under the conditions of Theorem A, there exists a constant l independent of s and i and polynomials $w_{n,i}$ (whose zeros lie on Δ_2), deg $w_{n,i} \geqslant n - r_i(s) - l$, $s \in \mathbb{N}$, $i = 1, 2, \ldots, m$, such that

$$\left[\frac{Q_n f_i - P_{n,i}}{w_{n,i}} \right](z) = O(z^{-n-1+l}) \in H(D). \tag{8}$$

In the sequel, the symbol $O(\cdot)$ always refers to the case when $z \to \infty$, and we will assume that the conditions of Theorem A are satisfied.

From (8) follows (see [1, §3, (1)]) that for all $i = 1, \ldots, m$ and $s \in \mathbb{N}$, the number of zeros Q_n in Δ_1 is at least equal to $n - l$. Moreover, if $S = \operatorname{supp} \mu_1$ and F is a finite union of k intervals such that $S \subset F \subset \Delta_1$, then the number of zeros of Q_n in F is at least equal to $n - l - k - 1$. Therefore, for each n, we can write $Q_n = Q_{n,1} Q_{n,2}$, where all zeros of $Q_{n,1}$ lie on F, deg $Q_{n,1} = n - l - k - 1$, deg $Q_{n,2} = l'(n) = l' \leqslant l + k + 1$, and $Q_{n,1}$, $Q_{n,2}$ are monic.

Take an arbitrary $x_0 \in S$ and put

$$G_{n,i}(z) = \frac{Q_{n,2}(z)}{(z - x_0)^{l'}} (f_i - R_{n,i}).$$

From Theorem A we have that

$$G_{n,i}(z) \to 0, \quad K \subset \Omega = \overline{\mathbb{C}} \setminus F, \ s \to \infty, \ i = 1, \ldots, m.$$

Therefore, the families of functions $\{G_{n,i}\}$, $n \in \mathbb{N}$, $i = 1, \ldots, m$, are normal in Ω.

Before proceeding, we introduce some notation. By $g_S(z, \zeta)$ we denote the (generalized) Green's function of S with pole at $\zeta \in D$. If cap $S = 0$, then $g_S(z, \zeta) \equiv \infty$ (for the concept of capacity and generalized Green's function see, e.g., [2]).

For arbitrary $K \subset \Omega$ put

$$\kappa = \kappa(K, F, \Delta_2) = \inf\{ g_F(z, \zeta);\ z \in K, \ \zeta \in \Delta_2 \}.$$

Clearly, $\kappa > 0$ for each $K \subset \Omega$. Fix $K \subset \Omega$. Consider a level line $\Gamma_\epsilon = \{z;\ g_F(z, \infty) = \epsilon\}$, where ϵ is so small that K and Δ_2 lie in the domain $\{z;\ g_F(z, \infty) > \epsilon\}$ (the exterior of Γ_ϵ). Put $\delta_\epsilon = \sup\{ g_F(z, \zeta);\ z \in \Gamma_\epsilon, \ \zeta \in \Delta_2 \}$. Obviously, $\delta_\epsilon \to 0$ as $\epsilon \to 0$.

Fix $i = 1, 2, \ldots, m$. Denote by $\zeta_{n,1}, \ldots, \zeta_{n,N(n)}$ the zeros of $w_{n,i}$. According to what was said above,

$$N(n) \geqslant n - r_i(s) - l. \tag{9}$$

On the other hand,

$$N(n) \leqslant n - r_i(s). \tag{10}$$

In fact, assume this is not so; that is, $N(n) > n - r_i(s)$. According to (2), (8) we would have

$$\left[\frac{Q_n f_i - P_{n,i}}{w'_{n,i}}\right](z) = \mathrm{O}(z^{-n-1}) \in H(D),$$

where $w'_{n,i}(z) = \prod_{k=1}^{n-r_i(s)}(z - \zeta_{n,k})$. That is, $R_{n,i}$ would be the nth multipoint Padé approximant which interpolates f_i at the $n - r_i(s)$ zeros of $w'_{n,i}$ and $n + r_i(s)$ times at infinity. Then (see, e.g., [5, (9)]),

$$(f_i - R_{n,i})(z) = \frac{w'_{n,i}(z)}{Q_n^2(z)} \int \frac{Q_n^2(x)}{w'_{n,i}(x)} \frac{f_{2,i}(x)\, \mathrm{d}\mu_1(x)}{(z-x)}.$$

Thus, $f_i - R_{n,i}$ may have on Δ_2 no other zeros than those in $w'_{n,i}$. On the other hand, in this case Q_n is the nth orthogonal polynomial with respect to the measure $f_{2,i}\, \mathrm{d}\mu_1 / w'_{n,i}$, and so all its zeros lie in Δ_1. Therefore,

$$\left[\frac{Q_n f_i - P_{n,i}}{w_{n,i}}\right](z) \notin H(D),$$

which contradicts (8). Hence, (10) holds.

Note that $G_{n,i} = \mathrm{O}(z^{-N'(n)})$, where $N'(n) = n + r_i(s) + 1 - l$. From this and (8) we have that the function

$$\psi_n(z) = \log |G_{n,i}(z)| + N'(n) g_F(z, \infty) + \sum_{k=1}^{N(n)} g_F(z, \zeta_{n,k})$$

is subharmonic in Ω, and since $\{G_{n,i}\}$, $n \in \mathbb{N}$, is normal in that region, then

$$\psi_n(z) \leqslant A_\epsilon + N'(n)\epsilon + N(n)\delta_\epsilon, \quad z \in \Gamma_\epsilon,$$

where A_ϵ is independent of n. By the maximum principle the same inequality holds in the exterior of Γ_ϵ, and in particular on K. Consequently,

$$\begin{aligned} |G_{n,i}(z)| &\leqslant \exp\left(A_\epsilon + N'(n)\epsilon + N(n)\delta_\epsilon - N'(n) g_F(z, \infty) - \sum_{k=1}^{N(n)} g_F(z, \zeta_{n,k})\right) \\ &\leqslant \exp(A_\epsilon + N'(n)(\epsilon - \tau) + N(n)(\delta_\epsilon - \kappa)), \quad z \in K, \end{aligned} \tag{11}$$

where $\tau = \tau(K, F) = \inf\{g_F(z, \infty);\ z \in K\}$.

Theorem 2. *Under the hypothesis of Theorem A, we have that for arbitrary*

$$\nu' < \nu = \frac{1}{2}\left(\frac{m+1}{m}\tau + \frac{m-1}{m}\kappa\right),$$

where $\tau = \tau(K, S)$ *and* $\kappa = \kappa(K, S, \Delta_2)$, *and each compact set* $K \subset \mathbb{C} \setminus S$,

$$\operatorname{cap}\{z \in K: |(f_i - R_{n,i})(z)| \geqslant \mathrm{e}^{-2n\nu'}\} \underset{s}{\to} 0. \tag{12}$$

Proof. Since from the hypothesis we have

$$\frac{n}{m} - c \leqslant r_i(s) \leqslant \frac{n}{m} + (m-1)c,$$

from (9), (10) we obtain the following bounds for $N(n)$:

$$n - \frac{n}{m} - (m-1)c - l \leqslant N(n) \leqslant n - \frac{n}{m} + c. \tag{13}$$

For $N'(n)$ we have

$$n + \frac{n}{m} + 1 - c - l \leqslant N'(n) \leqslant n + \frac{n}{m} + (m-1)c + 1 - l. \tag{14}$$

Fix $K \subset \mathbb{C} \setminus S$. Take F as above such that $F \cap K = \emptyset$. Using (11), (13) and (14), we obtain that for all sufficiently small $\epsilon > 0$,

$$\overline{\lim_{s}} \| G_{n,i}(z) \|_K^{1/2n} \leqslant \exp\left\{ \frac{1}{2} \left(\frac{m+1}{m}(\epsilon - \tau) + \frac{m-1}{m}(\delta_\epsilon - \kappa) \right) \right\},$$

with $\tau = \tau(K, F)$ and $\kappa = \kappa(K, F, \Delta_2)$. Letting $\epsilon \to 0$ and then making F "shrink down" to S, we arrive at

$$\overline{\lim_{s}} \| G_{n,i}(z) \|_K^{1/2n} \leqslant \exp\left\{ -\frac{1}{2} \left(\frac{m+1}{m}\tau + \frac{m-1}{m}\kappa \right) \right\}, \tag{15}$$

with $\tau = \tau(K, S)$ and $\kappa = \kappa(K, S, \Delta_2)$.

Note that if $\nu' < \nu'' < \nu$, then from (15), for all sufficiently large n,

$$\| G_{n,i}(z) \|_K \leqslant e^{-2n\nu''}, \quad n \geqslant n_0.$$

On the other hand,

$$\begin{aligned}
\{ z \in K: |(f_i - R_{n,i})(z)| \geqslant e^{-2n\nu'} \} &= \left\{ z \in K: |G_{n,i}(z)| \geqslant e^{-2n\nu'} \left| \frac{Q_{n,2}(z)}{(z - x_0)^{l'}} \right| \right\} \\
&\subset \left\{ z \in K: \left| \frac{Q_{n,2}(z)}{(z - x_0)^{l'}} \right| \leqslant e^{-2n(\nu'' - \nu')} \right\} \\
&\subset \{ z \in K: |Q_{n,2}(z)| \leqslant d e^{-2n(\nu'' - \nu')} \},
\end{aligned}$$

where $d = \sup\{ |z - x_0| : z \in K \}$. Therefore (see [2]),

$$\operatorname{cap}\{ z \in K: |(f_i - R_{n,i})(z)| \geqslant e^{-2n\nu'} \} \leqslant (d e^{-2n(\nu'' - \nu')})^{1/l'(n)}.$$

Since $l'(n) \leqslant l + k + 1$ independent of s, taking $s \to \infty$ we obtain (12). The proof of Theorem 2 is complete. □

Theorem 1 with $\nu = \frac{1}{2}\{(m+1)\tau/m + (m-1)\kappa/m\}$ is an immediate consequence of (15). It is sufficient to note that under those conditions all the zeros of Q_n lie in Δ_1. Thus, for all $z \in K \subset D$,

$$0 < A_1(K, F) \leqslant \left| \frac{Q_{n,2}(z)}{(z - x_0)^{l'}} \right| \leqslant A_2(K, F) < \infty, \quad s \in \mathbb{N}.$$

With this value of ν, it is obvious that when cap $S = 0$, then (7) (in Theorem 1) can be written as

$$\lim_{s\to\infty} \| f_i - R_{n,i} \|_K^{1/2n} = 0, \quad i = 1, \ldots, m.$$

This occurs, for example, if S is a numerable set.

That ν does not depend on i is to be expected since all the functions f_i are obtained integrating through μ_1, thus are very similar from the analytic point of view and they all receive approximately the same interpolation treatment $(r_i(s) \geqslant n/m - c)$.

Unfortunately, we are forced to give a very coarse value for the κ term because we have little information on how the zeros of $w_{n,i}$ are distributed on Δ_2. Nevertheless, if $\mu_1' > 0$ a.e. on Δ_1, it is not hard to observe (see [4]) that there exists a constant $\nu_1 = \nu_1(K, \Delta_1, \Delta_2) < \infty$ such that

$$\underline{\lim}_{s\to\infty} \| f_i - R_{n,i} \|_K^{1/2n} \geqslant \exp(-\nu_1), \quad i = 1, \ldots, m.$$

Therefore, the bound in Theorem 1 is correct in order. For such measures it should be possible to prove the existence of the limit in (7). In this case, the polynomials $w_{n,i}$ probably have some type of nth root asymptotics and then a more precise estimate for the κ term can be given (see, e.g., [5, Theorem 2] and [6, Theorem on p. 127]).

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